An Analysis of Five-copy Transmission with Majority Combining

Vladimir Vuković

Abstract — **In this paper, we analyze the five-copy transmission with bit error correction mechanism based on majority combining and derive the exact expressions for calculating the probability of successful single and double bit error correction using majority logic decision in the same bit positions. The results of analysis show that the majority combining offers much better performance compared to the simple selection combining techniques.**

Keywords — **Bit error correction, five-copy transmission, majority logic decision, transmission error probability.**

I. Introduction

IN order to improve the reliability of packet transmission in the presence of high bit error rate, authors have proposed simple hybrid error recovery procedures based on the multicopy transmission and majority combining technique that requires only minor modifications at the receiver. A *majority* combining scheme employs *majority decision* criterion in the same bit positions and verification of the frame checksum in the result copy [1]. This technique may be especially interesting in cases where applications already support multicopy transmission.

Our goal is to derive the exact expressions for transmission error probability and probability of successful single and double error correction using the majority logic in the same bit positions. The analysis is presented in the case of five-copy transmission with majority logic decision assuming the following: (a) frame synchronization is perfect, i.e., there are no problems in recognizing the beginning and in identifying each frame, (b) errors are statistically independent and uniformly distributed, and (c) frame checksum bits are transmitted without errors.

The paper is organized as follows. Sections II deals with theoretical analysis of the transmission error probability. The theoretical results are reported in Section III. Conclusions are provided in Section IV. The exact calculation of the probability of successful single and double bit error correction using majority logic in the same bit positions is given in the Appendix.

Vladimir Vuković, National Employment Service, Gundulićev venac 23–25, 11 000 Belgrade, Serbia (phone: 381-64-8107043, fax: 381-11-3209225, e-mail: vvukovic@nsz.sr.gov.yu).

II. Transmission Error Probability

The analysis of multicopy transmission is based on the simplified model shown in Fig. 1. The transmitter (side A) sends to the receiver (side B) the blocks that consist of five identical frame copies. Each copy of a frame contains user packet (I) and header (H) with additional bits for error detection, numbering, and identification. The transmission is considered successful if at least one of five received copies contains no detectable errors or if the implementation of bit-by-bit majority combining procedure yields frame in which all errors are corrected. In case when there is at least one bit position with triple and more errors, further retransmission is required.

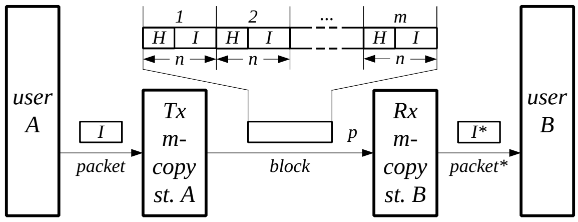

Fig. 1. Simplified model of multicopy scheme.

Transmission error probability of hybrid m-copy procedure with majority combining is equal [2]:

$$p_h(p,n,m) = p_m(p,n,m) - q_{ml}(p,n,m)\,. \qquad (1)$$

where p_m is probability that all m copies are transmitted with errors, and q_{ml} is correction factor (p denotes bit error probability, and n, frame size). We show in Appendix that in the case of single or double errors in the same bit positions, the expression for the correction factor is:

$$q_{ml}(p,n,m) = \sum_{m_1=0}^{\binom{m}{1}} \sum_{m_2=0}^{\binom{m}{2}} \alpha(m_1,m_2) \cdot \sum_{i=0}^{m_1}\sum_{j=0}^{m_2}(-1)^{i+j} \cdot \binom{m_1}{i}\binom{m_2}{j} \cdot [(1-p)^m + (m_1-i)\cdot p^1 \cdot (1-p)^{m-1} + (m_2-j)\cdot p^2 \cdot (1-p)^{m-2}]^n \qquad (2)$$

where coefficients m_1 and m_2 represent the number of various events with single and double error combinations, respectively, which provide at least one error in each copy, but not more than two errors in the same bit error positions. The number of events with identical pairs of coefficients m_1 and m_2 is denoted with $\alpha(m_1,m_2)$.

III. Numerical results

Transmission error probability p_m of standard and p_h of hybrid five-copy procedures with majority combining in the case of combination of single and double errors as function of bit error probability p is presented in Fig. 2. It can be observed that probabilities p_m and p_h increase monotonously with increasing of p. In addition to the analytical results, simulation results (marked with (*), (+) and (x)) are shown. For the adopted values of the bit error probability in the range 0.001 to 1, we generated error pattern of size n_f x n bits (n_f = 3000 copies, n = 128 bits). The agreement between analytical expression and simulation results is visible. When bit error probability is equal 0.02, transmission error probability of standard procedure is approximately 0.67, and hybrid procedure 0.01. The contribution of majority combining for the same bit error probability is 0.66.

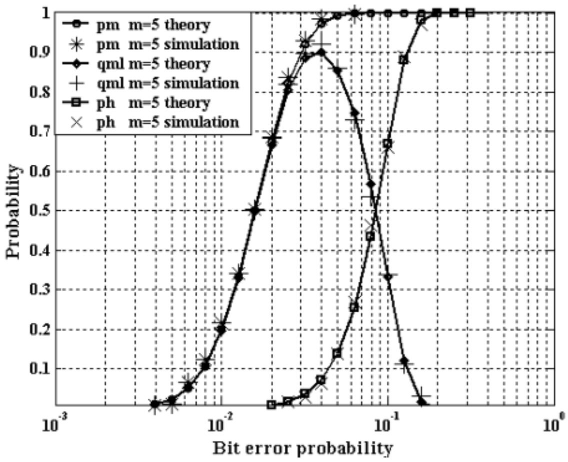

Fig. 2. Transmission error probability and contribution of five-copy majority combining.

Fig. 3 shows the relationship between bit error probability p and contribution of majority combining q_{ml} for different values of the frame size n. It can be observed that the contribution of the majority combining depends on the frame size and that there are bit error probabilities for which this contribution is maximal.

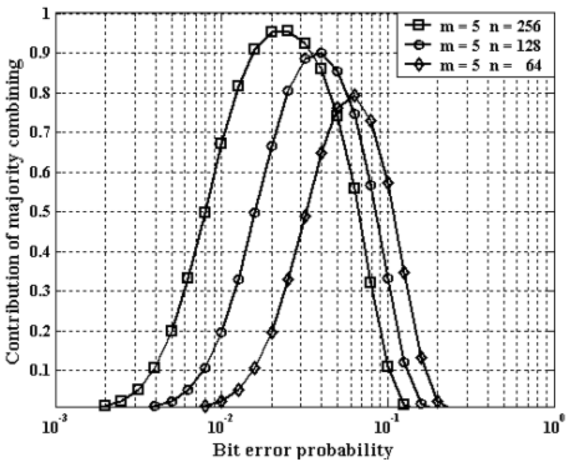

Fig. 3. Contribution of five-copy majority combining for different values of frame size.

IV. Conclusion

In this paper, we presented theoretical analysis of the five-copy transmission with error correction mechanism based on majority logic decision. We derived analytical expression for calculating transmission error probability and probability of successful single and double bit error correction using majority logic in the same bit positions. We showed that proposed multicopy scheme with majority combining offer better features comparing to the standard scheme.

Appendix

Methods for analyzing contribution of the majority logic in cases of three-copy and five-copy models are similar. Hence, three-copy analysis can be "conditionally" extended to cases with double errors in the same bit error positions [2]. Let S_i (i=0:31) represent possible combinations of errors that may occur in the transmission of bits at the same position of five copies of received frames. These outcomes can be specified by the j-th channel error indicators $E(j,i)$, j = 1, 2, 3, 4, and 5 as shown in Table 1.

Table 1: Specification of elementary events S_i.

Event S_i	Error indicator					$\sum$ Error
	E(1,i)	E(2,i)	E(3,i)	E(4,i)	E(5,i)	
0	0	0	0	0	0	0
1	1	0	0	0	0	1
2	0	1	0	0	0	1
3	0	0	1	0	0	1
4	0	0	0	1	0	1
5	0	0	0	0	1	1
6	1	1	0	0	0	2
7	1	0	1	0	0	2
8	1	0	0	1	0	2
9	1	0	0	0	1	2
10	0	1	1	0	0	2
11	0	1	0	1	0	2
12	0	1	0	0	1	2
13	0	0	1	1	0	2
14	0	0	1	0	1	2
15	0	0	0	1	1	2
16	1	1	1	0	0	3
17	1	1	0	1	0	3
18	1	1	0	0	1	3
19	1	0	1	1	0	3
20	1	0	1	0	1	3
21	1	0	0	1	1	3
22	0	1	1	1	0	3
23	0	1	1	0	1	3
24	0	1	0	1	1	3
25	0	0	1	1	1	3
26	1	1	1	1	0	4
27	1	1	1	0	1	4
28	1	0	1	1	1	4
29	1	1	0	1	1	4
30	0	1	1	1	1	4
31	1	1	1	1	1	5

*(1 denotes erroneous and 0 denotes correct bit transmission)

Assuming that error indicators $E(j,i)$ have value 1 if there was an error and value 0 in the opposite case, the probability p_i of the elementary event S_i is given by:

$$p_i = \prod_{j=1}^{5} p^{E(j,i)} \cdot q^{1-E(j,i)} . \qquad (3)$$

The transmission of five copies can be described as complex event where elementary event S_i is repeated $a_i \cdot k_i$ times. With a_i, we denoted the existence of event S_i and with k_i, the number of its occurrence. The probability of this complex event has multinomial distribution:

$$P(S_0 = a_0 \cdot k_0,\ S_1 = a_1 \cdot k_1, \cdots,\ S_{31} = a_{31} \cdot k_{31}) = n! \cdot \prod_{i=0}^{31} \frac{p_i^{a_i \cdot k_i}}{(a_i \cdot k_i)!} \qquad (4)$$

where $a_0 \cdot k_0 + a_1 \cdot k_1 + \cdots + a_{31} \cdot k_{31} = n$ and a_i = 1 or 0 (1 denotes the existence of event S_i).

For the error correction algorithm based on majority logic decision in the five-copy transmission, of interest are only cases of single or double errors in the same bit positions. All five copies are erroneous if there exists combination of events S_i (i=1:15) that provides at least one error in each copy, but no more than two errors in the same bit positions, while all other events can be only S_0. The identification of such combinations of elementary events S_i (i=1:15) that fulfills the above stated condition is a rather complicated problem. To solve this problem, let us first create a matrix A of size [2^{15}x15] whose elements $a_i = a_i^{(*)}$ denote either presence ($a_i = 1$) or absence ($a_i = 0$) of an elementary event S_i. Each row of this matrix represents one possible combination of event S_i (i=1:15). The total number of all possible combinations is 2^{15}.

$$A = \begin{bmatrix} A^{(0)} \\ \vdots \\ A^{(j)} \\ \vdots \\ A^{(2^{15}-1)} \end{bmatrix} = \begin{bmatrix} a_1^{(0)} & \cdots & a_5^{(0)} & | & a_6^{(0)} & \cdots & a_{15}^{(0)} \\ & & & \vdots & & & \\ a_1^{(j)} & \cdots & a_5^{(j)} & | & a_6^{(j)} & \cdots & a_{15}^{(j)} \\ & & & \vdots & & & \\ a_1^{(2^{15}-1)} & \cdots & a_5^{(2^{15}-1)} & | & a_6^{(2^{15}-1)} & \cdots & a_{15}^{(2^{15}-1)} \\ \leftarrow & m_1 & \rightarrow & | & \leftarrow & m_2 & \rightarrow \end{bmatrix}$$

$$= \begin{bmatrix} 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \\ \vdots \\ 1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1 \end{bmatrix} \qquad (5)$$

Next, let us create a matrix S of size [15x5] whose elements represent elementary events S_1, S_2, …, S_{15}.

$$S = \begin{bmatrix} S_1 \\ \vdots \\ S_5 \\ S_6 \\ \vdots \\ S_{15} \end{bmatrix} = \begin{bmatrix} 1\,0\,0\,0\,0 \\ \vdots \\ 0\,0\,0\,0\,1 \\ 1\,1\,0\,0\,0 \\ \vdots \\ 0\,0\,0\,1\,1 \end{bmatrix} \qquad (6)$$

Elements of the first column of matrix S represent either presence or absence of errors in the first copy. Elements of second column represent either presence or absence of errors in the second copy, and so on. Finally, elements in last column represent either presence or absence of errors in the fifth copy of the frame.

Let R denote the product of matrix A and matrix S. A criteria of majority logic decision is satisfied if all elements of a j-th row of matrix R are equal or greater than 1. In that case, for j-th rows of matrix A, we determine coefficients m_1 and m_2 as $m_1 = a_1 + a_2 + \cdots + a_5$ and $m_2 = a_6 + a_7 + \cdots + a_{15}$. These coefficients represent the number of various events with single and double error combinations, respectively. In Table 2, all possible combinations of events with single and double errors that fulfill above condition are illustrated. We use $\alpha(m_1, m_2)$ to denote the number of events with same pairs of coefficients m_1 and m_2.

TABLE 2: SPECIFICATION OF $\alpha(m_1, m_2)$ COEFFICIENTS.

m1 / m2	0	1	2	3	4	5
0	0	0	0	0	0	1
1	0	0	0	10	20	10
2	0	15	90	180	150	45
3	30	230	630	810	500	120
4	135	750	1650	1800	975	210
5	222	1140	2340	2400	1230	252
6	205	1030	2070	2080	1045	210
7	120	600	1200	1200	600	120
8	45	225	450	450	225	45
9	10	50	100	100	50	10
10	1	5	10	10	5	1

The total correction factor q_{ml} represents the sum of partial correction factors q^*_{ml} for each pair of coefficients (m_1, m_2) shown in Table II. In the case $m_1 = 5$ and $m_2 = 10$, expression for the partial correction factor q^*_{ml} is:

$$\begin{aligned} q^*_{ml} = G(&a_0 \cdot k_0 \geq 0, a_1 \cdot k_1 > 0, \cdots, a_5 \cdot k_5 > 0, \\ &a_6 \cdot k_6 > 0, a_7 \cdot k_7 > 0, \cdots, a_{15} \cdot k_{15} > 0, \\ &a_{16} \cdot k_{16} = 0, a_{17} \cdot k_{17} = 0, \cdots, a_{31} \cdot k_{31} = 0,) \end{aligned} \qquad (7)$$

We now introduce following notations:

$$c_i = a_i \cdot k_i , \qquad (8)$$

$$G(c_0, c_1, \cdots, c_{15}, c_{16} = 0, \cdots, c_{31} = 0) \equiv G(c_0, c_1, \cdots, c_{15}), \qquad (9)$$

$$G(c_i \geq 0) \equiv G(c_i), G(c_i = 0) \equiv G(c_i^0), G(c_i > 0) \equiv G(c_i^+) . \qquad (10)$$

We can derive analytical expression for partial correction factor q^*_{ml} in the case $m_1 = 3$ and $m_2 = 3$ (an example: $a_1 = a_2 = 0$, $a_3 = a_4 = a_5 = 1$, $a_6 = a_7 = a_8 = 1$, $a_9 = a_{10} = a_{11} = a_{12} = a_{13} = a_{14} = a_{15} = 0$). According to the [2], expression for the partial correction factor q^*_{ml} is:

$$
\begin{aligned}
G(c_0,c_3^+,c_4^+,c_5^+,c_6^+,c_7^+,c_8^+) &= G(c_0,c_3,c_4,c_5,c_6,c_7,c_8)- \\
&-G(c_0,c_3^0,c_4,c_5,c_6,c_7,c_8)-G(c_0,c_3,c_4^0,c_5,c_6,c_7,c_8)- \\
&-G(c_0,c_3,c_4,c_5^0,c_6,c_7,c_8)-G(c_0,c_3,c_4,c_5,c_6^0,c_7,c_8)- \\
&-G(c_0,c_3,c_4,c_5,c_6,c_7^0,c_8)-G(c_0,c_3,c_4,c_5,c_6,c_7,c_8^0)+ \\
&+G(c_0,c_3^0,c_4^0,c_5,c_6,c_7,c_8)+G(c_0,c_3^0,c_4,c_5^0,c_6,c_7,c_8)+ \\
&+G(c_0,c_3,c_4^0,c_5^0,c_6,c_7,c_8)+G(c_0,c_3^0,c_4,c_5,c_6^0,c_7,c_8)+ \\
&+G(c_0,c_3^0,c_4,c_5,c_6,c_7^0,c_8)+G(c_0,c_3^0,c_4,c_5,c_6,c_7,c_8^0)+ \\
&+G(c_0,c_3,c_4^0,c_5,c_6^0,c_7,c_8)+G(c_0,c_3,c_4^0,c_5,c_6,c_7^0,c_8)+ \\
&+G(c_0,c_3,c_4^0,c_5,c_6,c_7,c_8^0)+G(c_0,c_3,c_4,c_5^0,c_6^0,c_7,c_8)+ \\
&+G(c_0,c_3,c_4,c_5^0,c_6,c_7^0,c_8)+G(c_0,c_3,c_4,c_5^0,c_6,c_7,c_8^0)+ \\
&+G(c_0,c_3,c_4,c_5,c_6^0,c_7^0,c_8)+G(c_0,c_3,c_4,c_5,c_6^0,c_7,c_8^0)+ \\
&+G(c_0,c_3,c_4,c_5,c_6,c_7^0,c_8^0)- \\
&-G(c_0,c_3^0,c_4^0,c_5^0,c_6,c_7,c_8)-G(c_0,c_3^0,c_4^0,c_5,c_6^0,c_7,c_8)- \\
&-G(c_0,c_3^0,c_4^0,c_5,c_6,c_7^0,c_8)-G(c_0,c_3^0,c_4^0,c_5,c_6,c_7,c_8^0)- \\
&-G(c_0,c_3^0,c_4,c_5^0,c_6^0,c_7,c_8)-G(c_0,c_3^0,c_4,c_5^0,c_6,c_7^0,c_8)- \\
&-G(c_0,c_3^0,c_4,c_5^0,c_6,c_7,c_8^0)-G(c_0,c_3,c_4^0,c_5^0,c_6^0,c_7,c_8)- \\
&-G(c_0,c_3,c_4^0,c_5^0,c_6,c_7^0,c_8)-G(c_0,c_3,c_4^0,c_5^0,c_6,c_7,c_8^0)- \\
&-G(c_0,c_3^0,c_4,c_5,c_6^0,c_7^0,c_8)-G(c_0,c_3^0,c_4,c_5,c_6^0,c_7,c_8^0)- \\
&-G(c_0,c_3^0,c_4,c_5,c_6,c_7^0,c_8^0)-G(c_0,c_3,c_4^0,c_5,c_6^0,c_7^0,c_8)- \\
&-G(c_0,c_3,c_4^0,c_5,c_6^0,c_7,c_8^0)-G(c_0,c_3,c_4^0,c_5,c_6,c_7^0,c_8^0)- \\
&-G(c_0,c_3,c_4,c_5^0,c_6^0,c_7^0,c_8)-G(c_0,c_3,c_4,c_5^0,c_6^0,c_7,c_8^0)- \\
&-G(c_0,c_3,c_4,c_5^0,c_6,c_7^0,c_8^0)-G(c_0,c_3,c_4,c_5,c_6^0,c_7^0,c_8^0)+ \\
&+G(c_0,c_3^0,c_4^0,c_5^0,c_6^0,c_7,c_8)+G(c_0,c_3^0,c_4^0,c_5^0,c_6,c_7^0,c_8)+ \\
&+G(c_0,c_3^0,c_4^0,c_5^0,c_6,c_7,c_8^0)+G(c_0,c_3^0,c_4^0,c_5,c_6^0,c_7^0,c_8)+ \\
&+G(c_0,c_3^0,c_4^0,c_5,c_6^0,c_7,c_8^0)+G(c_0,c_3^0,c_4^0,c_5,c_6,c_7^0,c_8^0)+ \\
&+G(c_0,c_3^0,c_4,c_5^0,c_6^0,c_7^0,c_8)+G(c_0,c_3^0,c_4,c_5^0,c_6^0,c_7,c_8^0)+ \\
&+G(c_0,c_3^0,c_4,c_5^0,c_6,c_7^0,c_8^0)+G(c_0,c_3,c_4^0,c_5^0,c_6,c_7^0,c_8^0)+ \\
&+G(c_0,c_3,c_4^0,c_5,c_6^0,c_7^0,c_8^0)+G(c_0,c_3,c_4,c_5^0,c_6^0,c_7^0,c_8^0)+ \\
&+G(c_0,c_3^0,c_4,c_5,c_6^0,c_7^0,c_8^0)+G(c_0,c_3,c_4^0,c_5^0,c_6^0,c_7^0,c_8)+ \\
&+G(c_0,c_3,c_4^0,c_5^0,c_6^0,c_7,c_8^0)- \\
&-G(c_0,c_3^0,c_4^0,c_5^0,c_6^0,c_7^0,c_8)-G(c_0,c_3^0,c_4^0,c_5^0,c_6^0,c_7,c_8^0)- \\
&-G(c_0,c_3^0,c_4^0,c_5^0,c_6,c_7^0,c_8^0)-G(c_0,c_3^0,c_4^0,c_5,c_6^0,c_7^0,c_8^0)- \\
&-G(c_0,c_3^0,c_4,c_5^0,c_6^0,c_7^0,c_8^0)-G(c_0,c_3,c_4^0,c_5^0,c_6^0,c_7^0,c_8^0)+ \\
&+G(c_0,c_3^0,c_4^0,c_5^0,c_6^0,c_7^0,c_8^0)
\end{aligned}
\tag{11}
$$

Substituting (3) in (11) and using the known characteristics of multinomial distribution, expression for q^*_{ml} becomes:

$$
\begin{aligned}
q^*_{ml} &= G(c_0,c_3^+,c_4^+,c_5^+,c_6^+,c_7^+,c_8^+) = \\
&= \binom{m_1}{0}\binom{m_2}{0}[q^5+(m_1-0)\cdot p^1q^4+(m_2-0)\cdot p^2q^3]^n - \\
&-\binom{m_1}{1}\binom{m_2}{0}[q^5+(m_1-1)\cdot p^1q^4+(m_2-0)\cdot p^2q^3]^n - \\
&-\binom{m_1}{0}\binom{m_2}{1}[q^5+(m_1-0)\cdot p^1q^4+(m_2-1)\cdot p^2q^3]^n + \\
&+\binom{m_1}{2}\binom{m_2}{0}[q^5+(m_1-2)\cdot p^1q^4+(m_2-0)\cdot p^2q^3]^n + \\
&+\binom{m_1}{1}\binom{m_2}{1}[q^5+(m_1-1)\cdot p^1q^4+(m_2-1)\cdot p^2q^3]^n + \\
&+\binom{m_1}{0}\binom{m_2}{2}[q^5+(m_1-0)\cdot p^1q^4+(m_2-2)\cdot p^2q^3]^n - \\
&-\binom{m_1}{3}\binom{m_2}{0}[q^5+(m_1-3)\cdot p^1q^4+(m_2-0)\cdot p^2q^3]^n - \\
&-\binom{m_1}{2}\binom{m_2}{1}[q^5+(m_1-2)\cdot p^1q^4+(m_2-1)\cdot p^2q^3]^n - \\
&-\binom{m_1}{1}\binom{m_2}{2}[q^5+(m_1-1)\cdot p^1q^4+(m_2-2)\cdot p^2q^3]^n - \\
&-\binom{m_1}{0}\binom{m_2}{3}[q^5+(m_1-0)\cdot p^1q^4+(m_2-3)\cdot p^2q^3]^n + \\
&+\binom{m_1}{3}\binom{m_2}{1}[q^5+(m_1-3)\cdot p^1q^4+(m_2-1)\cdot p^2q^3]^n + \\
&+\binom{m_1}{2}\binom{m_2}{2}[q^5+(m_1-2)\cdot p^1q^4+(m_2-2)\cdot p^2q^3]^n + \\
&+\binom{m_1}{1}\binom{m_2}{3}[q^5+(m_1-1)\cdot p^1q^4+(m_2-3)\cdot p^2q^3]^n - \\
&-\binom{m_1}{3}\binom{m_2}{2}[q^5+(m_1-3)\cdot p^1q^4+(m_2-2)\cdot p^2q^3]^n - \\
&-\binom{m_1}{2}\binom{m_2}{3}[q^5+(m_1-2)\cdot p^1q^4+(m_2-3)\cdot p^2q^3]^n + \\
&+\binom{m_1}{3}\binom{m_2}{3}[q^5+(m_1-3)\cdot p^1q^4+(m_2-3)\cdot p^2q^3]^n .
\end{aligned}
\tag{12}
$$

Since $p+q=1$, expression (12) can be represented as a double sum:

$$
\begin{aligned}
q^*_{ml}(p,n,m_1,m_2) &= \sum_{i=0}^{m_1}\sum_{j=0}^{m_2}(-1)^{i+j}\cdot\binom{m_1}{i}\binom{m_2}{j} \\
&\cdot[(1-p)^5+(m_1-i)\cdot p^1\cdot(1-p)^4+(m_2-j)\cdot p^2\cdot(1-p)^3]^n .
\end{aligned}
\tag{13}
$$

Adding the partial probabilities $q^*_{ml}(p,n,m_1,m_2)$ by all pairs of coefficients m_1 and m_2 from Table 2, the final expression for total correction factor in case of the five-copy transmission (m = 5) is:

$$
\begin{aligned}
q_{ml}(p,n,5) &= \sum_{m_1=0}^{5}\sum_{m_2=0}^{10}\alpha(m_1,m_2)\cdot\sum_{i=0}^{m_1}\sum_{j=0}^{m_2}(-1)^{i+j}\cdot\binom{m_1}{i}\binom{m_2}{j} \\
&\cdot[(1-p)^5+(m_1-i)\cdot p^1\cdot(1-p)^4+(m_2-j)\cdot p^2\cdot(1-p)^3]^n .
\end{aligned}
\tag{14}
$$

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