Optimization of Polynomial-based Interpolation Filters using Discrete-time Model

Đorđe Babić

Abstract — **Polynomial-based interpolation filters provide flexible means in many digital signal processing applications, for example in arbitrary and asynchronous sampling rate conversion, timing adjustments and symbol synchronizations in digital receivers, image resizing and rotation etc. The impulse response of the polynomial-based interpolation filter is represented as continuous-time piecewise polynomial function. These filters have efficient implementation form known in Literature as Farrow structure, which has several modifications. The polynomial-based interpolator filters has been optimized using hybrid analogue/digital model. However, in practice these filters are implemented in digital system and they have corresponding FIR equivalent filter. This paper presents methods for discrete-time optimization of polynomial-based interpolation filters. These methods are based on relation between polynomial-based interpolation filter and corresponding FIR model filter. This paper also compares properties of the filters optimized in traditional way using continuous-time model, and filter optimized using discrete-time model.**

Keywords — **FIR filter, polynomial-based filter**

I. INTRODUCTION

POLYNOMIAL-based interpolation filters provide flexible means in many digital signal processing applications, for example in arbitrary and asynchronous sampling rate conversion, timing adjustments and symbol synchronizations in digital receivers, image resizing and rotation etc. The impulse response $h_a(t)$ of polynomial-based interpolation filter has the following properties [1], [2]. First, $h_a(t)$ is nonzero only in the interval $\leq t < NT$ with N being an even integer. Second, in each subinterval $nT \leq t < (n+1)T$ for n [illegible]1, …, N–2 $h_a(t)$ is expressible as a polynomial of the given low order M. Third, $h_a(t)$ is symmetric around $t = NT/2$ to guarantee the phase linearity of the resulting overall system. The length of polynomial segments T can be selected to be equal to the input or output sampling interval, a fraction of the input or output sampling interval, or an integer multiple of the input or output sampling interval. The advantage of mimicking the above system lies in the fact that the actual implementation can be efficiently performed by using the Farrow structure [3] or its modifications [4], [6]. The main advantage of the Farrow structure lies in the fact that it consists of fixed finite-impulse response (FIR) filters and there is only one changeable parameter being the so-called fractional interval μ. Besides this, the control of μ is easier during the operation than in the corresponding coefficient memory implementations, and the resolution of μ is limited only by the precision of arithmetic and not by the size of the memory. These characteristics of the Farrow structure make it a very attractive structure to be implemented using a VLSI circuit or a signal processor [2].

Sampling rate conversion (SRC) is utilized in many DSP applications where two signals or systems having different sampling rates are to be interconnected. The SRC factor in general can be an integer, one divided by an integer, a ratio of two integers, or an irrational number. The SRC factor is determined by

$$R = F_{out} / F_{in} = T_{in} / T_{out} , \tag{1}$$

where $F_{in} = 1/T_{in}$ and $F_{out} = 1/T_{out}$ are the original input sampling rate and the sampling rate after the conversion, respectively. The sampling rate conversion can be divided into two general cases. For $R < 1$, the sampling rate is reduced and this process is known as decimation. For $R > 1$, the sampling rate is increased and this process is known as interpolation [3].

In practical realizations, the SRC factor R is implemented as a ratio of two relatively prime integers, i.e., $R=L/K$. This means that in practice an irrational factor is not achievable, thus it is approximated by a rational number which is determined by the precision of the arithmetic used. In [6], a discrete time model was derived for the zeroth-order polynomial interpolation. In [8], the discrete-time modeling of the rational SRC is extended to the polynomial-based interpolators of arbitrary order. Further, in [8] it is shown that for any modification of the Farrow structure there exists a discrete-time FIR model filter.

This paper presents methods for discrete-time optimization of polynomial-based interpolation filters. These methods are based on relation between polynomial-based interpolation filter and corresponding FIR model filter. This paper also compares properties of the filters optimized in traditional way using continuous-time model, and filter optimized using discrete-time model.

II. HYBRID ANALOGUE/DIGITAL MODEL FOR SRC

In the most general case, the SRC can be regarded as a process of resampling an analogue signal according to the

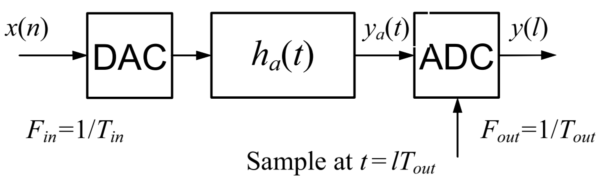

Fig. 1. Hybrid analogue/digital model to be mimicked in order to generate an efficient digital system for sampling rate conversion.

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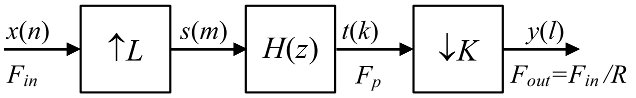

Fig. 2. Discrete-time model for SRC by rational factor.

hybrid analogue/digital model shown in Fig. 1 [3]. In this system, the discrete-time input sequence $x(n)$ with the sampling rate equal to $F_{in}=1/T_{in}$ is first converted into an analog continuous-time signal $x_s(t)$ using an ideal digital-to-analog converter (DAC). The resulting signal is the following sum of the weighted and shifted impulses:

$$x_s(t) = \sum_{k=-\infty}^{\infty} x(k)\delta_a(t-kT_{in}), \tag{2}$$

where $\delta_a(t)$ is the analogue Dirac delta function. This sequence is then filtered using an analogue filter with the impulse response $h_a(t)$ to generate the continuous-time signal given by

$$y_a(t) = \sum_{k=-\infty}^{\infty} x(k)h_a(t-kT_{in}). \tag{3}$$

Finally, the output $y_a(t)$ is sampled at the time instants $t=lT_{out}$ using the ideal analog-to-digital converter (ADC) to produce the following discrete-time output sequence $y(l)$:

$$y(l) = y_a(lT_{out}) = \sum_{k=-\infty}^{\infty} x(k)h_a(lT_{out}-kT_{in}). \tag{4}$$

It should be pointed out that in the decimation case, the filter with the impulse response $h_a(t)$ acts as an anti-aliasing filter rejecting the frequency components aliasing onto the new baseband. On the other hand, in the interpolation case, the role of this filter is to preserve the original baseband region and to eliminate the imaging components. Hence, it acts as an anti-imaging filter.

III. Discrete-time Model for Rational SRC

This part overviews the discrete-time model for the SRC. The model is valid for the case when the SRC factor is a ratio of two relatively prime integers, i.e., $R=L/K$.

When using the hybrid analogue/digital model directly for system analysis, aliasing may become a problem. Because of that, frequencies far higher than F_{in} and F_{out} must be taken into consideration, leading to a high computational complexity. It may also be difficult to determine how high frequencies should be included in the analysis in order to keep the approximation error, caused by the ignored aliases, at an acceptable level. The aliasing problem can be avoided by using the discrete-time model shown in Fig. 2. This is used earlier in [6] to model the zeroth order interpolation.

Sampling rate conversion by $R=L/K$ can be implemented as a cascade of interpolation by L and decimation by K. As shown in Fig. 2, the interpolation and decimation filters can be combined into a single filter, $H(z)$. In the general case, the sampling rate is first increased by L in the upsampler block. After that, the signal is filtered with transfer function $H(z)$ and, finally, the sampling rate is reduced by K using a downsampler block. The role of the filter $H(z)$ is to perform both aliasing and imaging attenuation. The key idea for the derivation of the relation

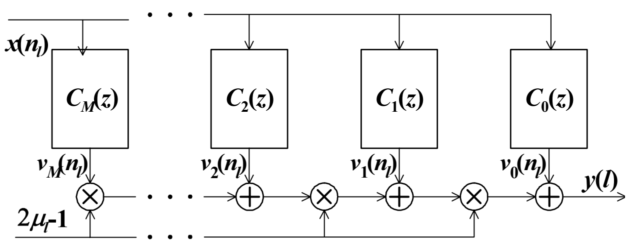

Fig. 3. Modified Farrow structure.

between the impulse response of the analogue filter $h_a(t)$, from hybrid model of Fig. 1, and discrete-time filter $H(z)$, in the discrete-time model of Fig. 2, is to sample the continuous-time impulse response $h_a(t)$ at the intermediate rate $F_p=LF_{in}=KF_{out}$. This sampling rate does not exist physically in a polynomial interpolator. In the discrete-time model, the period of the frequency response of the FIR filter $H(e^{j\omega})$ is F_p, that is, all frequency domain information is always contained within a finite band.

IV. Relation between Polynomial-based Interpolators and FIR Filters

This section explains the relation between modified Farrow structure of the polynomial based filters and discrete-time polyphase FIR model. This type of relation for other modifications of the Farrow structure can be found in [8].

Consider the system of Fig. 3, where the input sampling period is equal to T_{in}. The structure is known as the modified Farrow structure, and it is represented by [1], [2]:

$$y(l) = \sum_{n=0}^{N-1}\sum_{m=0}^{M} x(n_l - N/2 + n)c_m(n)(2\mu_l - 1)^m. \tag{5}$$

where

$$n_l = \lfloor l/R + \varepsilon \rfloor, \text{ and } \mu_l = l/R + \varepsilon - \lfloor l/R + \varepsilon \rfloor. \tag{6}$$

Here, N is the number of polynomial segments in $h_a(t)$, M is the order of polynomials, μ_l is the fractional interval, and ε is an offset value in the computation of μ_l. In the case of the rational sampling rate conversion factor $R=L/K$, the fractional interval μ_l has a certain (finite) number of values. It can be shown that in this case, μ_l is a periodic function, having a period of L. It can be concluded that the modified Farrow structure has an equivalent FIR representation. The equivalent FIR filter can be divided into L polyphase branches. Each polyphase branch corresponds to one value of the fractional interval μ_l. The overall transfer function of the equivalent polyphase FIR filter is given by

$$H(z) = \sum_{l=0}^{L-1} z^{-l} G_l(z^L), \tag{7}$$

where

$$G_l(z^L) = \sum_{n=0}^{N-1} g_l(n) z^{nL}. \tag{8}$$

The coefficients of the polyphase branches are computed from the coefficients of the original modified Farrow structure as follows:

$$g_l(n) = \sum_{m=0}^{M} c_m(n)(2\mu_l - 1)^m, \tag{9}$$

for l=0, 1, …, L−1, and n=0, 1,…, N−1. The relation between two sets of coefficients can be expressed in matrix form as:

$$\begin{bmatrix} g_0(n) \\ g_1(n) \\ \vdots \\ g_{L-1}(n) \end{bmatrix} = \begin{bmatrix} 1 & (2\mu_0-1) & \cdots & (2\mu_0-1)^M \\ 1 & (2\mu_1-1) & \cdots & (2\mu_1-1)^M \\ \vdots & \vdots & \cdots & \vdots \\ 1 & (2\mu_{P-1}-1) & \cdots & (2\mu_{P-1}-1)^M \end{bmatrix} \cdot \begin{bmatrix} c_0(n) \\ c_1(n) \\ \vdots \\ c_M(n) \end{bmatrix}, \quad (10)$$

for n=0, 1,…, N−1. Finally,

$$\begin{bmatrix} \mathbf{G}_0 \\ \mathbf{G}_1 \\ \vdots \\ \mathbf{G}_{L-1} \end{bmatrix} = \begin{bmatrix} 1 & (2\mu_0-1) & \cdots & (2\mu_0-1)^M \\ 1 & (2\mu_1-1) & \cdots & (2\mu_1-1)^M \\ \vdots & \vdots & \cdots & \vdots \\ 1 & (2\mu_{L-1}-1) & \cdots & (2\mu_{L-1}-1)^M \end{bmatrix} \cdot \mathbf{C}^{\mathbf{T}}. \quad (11)$$

Here $\mathbf{G}_l$, for l=0, 1,…, L−1, represents the row vector of polyphase branch coefficients. We conclude that in the case of the sampling rate conversion by a rational factor, it is equivalent to implement the required filter either by using the Farrow structure, or the corresponding polyphase FIR filter. The equivalent polyphase FIR representation using commutators is shown in Fig. 4. The commutator switches from one polyphase branch output to another according to cyclic pattern, jumping with a step length of K branches. This corresponds to decimation in the discrete-time model. The polyphase FIR branches work at the input sampling rate F_{in}, and the commutator switches at the output sampling rate F_{out}. The overall structure contains L polyphase branches of length N, where N is the length of the analogue polynomial-based filter (subfilter length in the Farrow structure). Thus, the overall number of multipliers is NL. The FIR model filter can be made symmetric if initial value of fractional interval (offset value) is ε=1/(2L). In this case, the number of multipliers in FIR model filter is equal to NL/2.

V. Design using discrete-time model

Relation between coefficients of polynomial-based filter and its FIR equivalent can be effectively used for the optimization of the filter coefficients directly in digital domain. For that purpose, Eq. (11) can be written in following form:

$$\mathrm{C}^{\mathrm{T}} = \begin{bmatrix} 1 & (2\mu_0-1) & \cdots & (2\mu_0-1)^M \\ 1 & (2\mu_1-1) & \cdots & (2\mu_1-1)^M \\ \vdots & \vdots & \cdots & \vdots \\ 1 & (2\mu_{L-1}-1) & \cdots & (2\mu_{L-1}-1)^M \end{bmatrix}^{-1} \cdot \begin{bmatrix} \mathrm{G}_0 \\ \mathrm{G}_1 \\ \vdots \\ \mathrm{G}_{L-1} \end{bmatrix}. \quad (12)$$

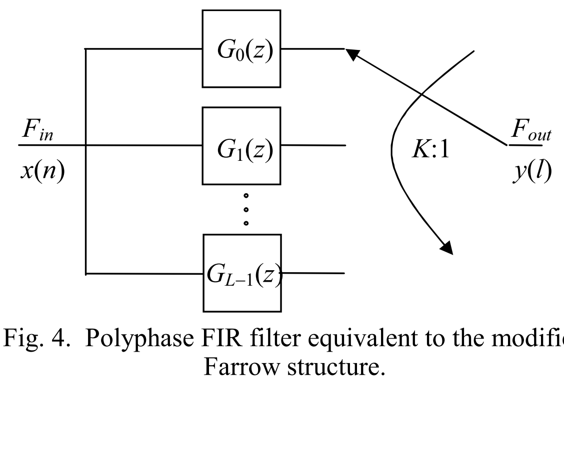

Fig. 4. Polyphase FIR filter equivalent to the modified Farrow structure.

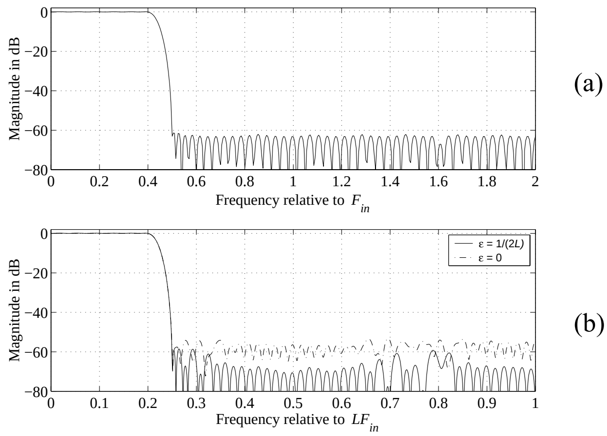

Fig 5. Frequency responses: (a) polynomial-based filter CT design, (b) FIR filter DT design.

Based on Eq. (12), the design method is straightforward. The first step is to optimize FIR filter coefficients $h(n)$ using desirable requirements with frequency scaled by L, and suitable optimization method, for example Remez. Then, it is necessary to perform polyphase decomposition of the filter $H(z)$, in order to determine the row vectors of polyphase branch coefficients G_l, for l=0, 1, …L-1. Finally, the last step is to calculate coefficients of the polynomial-based filter according to Eq. (12). By inspecting Eq. (12), it is obvious that the expression can always be solved if L=M+1. Thus, the length of starting FIR filter $H(z)$ is N_{FIR}=N*L=N*(M+1).

A. Comparison of Continuous-time and Discrete-Time model for design of polynomial-based structure.

For purposes of this section, we define L_T as the target value of L, i.e., the value used when designing the filter. The value of L used in the realization is denoted by L_R. For comparison, a set of interpolators were optimized for different signal bandwidths, attenuation specifications, and values of L_T. In each case, the same subfilter lengths and polynomial degrees were used for the CT design and all target conversion ratios of the DT model. It can be observed that the peak ripple depends on LR. When L_R approaches infinity, the DT ripple level approaches that of the CT frequency response, whereas the maximum attenuation loss is suffered at the lowest values of LR When interpolation filters are optimized using the DT model, the peak ripple level of the CT frequency response may become greater than that of the DT frequency response at L_R=L_T. If L_R is then changed, the higher ripples of the CT response become visible because imaging or aliasing changes. The DT model is easy to handle because the frequency response can be computed numerically from the impulse response. The CT model, on the other hand, requires a more analytic approach, and the formulae for the frequency response are rather complicated [2]. Furthermore, numerical sensitivity problems occur and must be overcome in the CT frequency response. On the other hand, numerical sensitivity issues can be expected to arise also in the DT model if L is large.

For DT design, the Farrow coefficients have full control of impulse response when M=L-1. However, in the most cases in practice L is significantly larger than M, and we cannot use N_{FIR}=N*L, for the length of starting FIR filter. The main reason is saturation threshold for M, after which it is not possible to obtain improvement of frequency characteristics of the Farrow filter. In these cases, it is

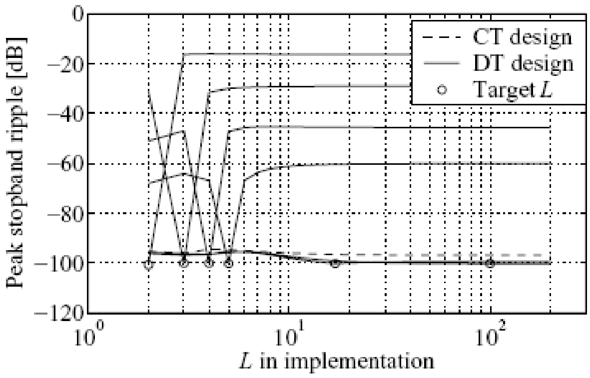

Fig 6. Stopband attenuation: *f*p=0.8, continuous stopband, *N*=14, *M*=min{*L*R–1,4}.

desirable to use the threshold value of *M*, and thus starting FIR filter has length $N_{FIR}=N*(M+1)$. The remaining task is to find a method for accurate estimation of threshold value of *M*, for given design requirements.

VI. Case Studies

Let us consider an integer interpolation by 4 using the modified Farrow structure. In this case $R=L/K=4$, thus there are $L=4$ polyphase branches (corresponding to L different values of μ_l). The requirements for the interpolation filter are as follow: stopband attenuation at least $A_s=60$ dB, passband distortion at most $A_p=0.001$, passband edge $f_p=0.4*F_{in}$, and stopband edge $f_s=0.5*F_{in}$. These requirements are met by using the modified Farrow structure having $N=28$ and $M=4$.

We can design an FIR filter satisfying the same requirements as in previous case, and then using the Eq. (11), we can convert it into the modified Farrow structure. The length of designed filter is $N_{FIR}=N*L=112$. Figure 5 shows frequency response of the polynomial-based filter designed using continuous time model (Fig. 5.(a)), and polynomial-based filter designed using discrete time model (Fig. 5.(b)).

The peak stopband ripple levels of the DT frequency responses of a few example filters are shown in Figs. 6 as functions of L_R. In the plots, M and N denote the polynomial degree and number of polynomial segments of the filter, respectively. The passband edge *f*p is normalized so that unity corresponds to half of the sample rate in the subfilters. Only minimax optimization was considered in the comparison

VII. Conclusions

It has been shown that in the case of SRC by a rational factor, for any polynomial-based filter of arbitrary order there exists an equivalent time-invariant FIR filter. This discrete-time filter can be used in filter design and distortion analysis to overcome the aliasing problems associated with the hybrid analog/digital model.

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