REVIEW OF LASER BEAM TRANSFORMATION THROUGH SPACE AND LENSES

Aleksandar Marinčić

Abstract. *In transmission of laser beam through lossless space or through lenses or non-uniform media, the beam parameters change. If paraxial approximation is applicable, many problems in beam propagation through various media conditions can be reduced to closed form solutions. In this paper we consider transformation of circular and elliptical laser beam of Gaussian shape in passing through uniform medium or through lens or system of lenses.*

***Key words**: Laser circular and elliptical beams, lenses, beam parameters.*

1. INTRODUCTION

Calculation of laser radiation field can be made using the aperture field method or using the spectrum of plane waves [1], or paraxial approximation of Helmholtz equation [2]. All these approaches lead to the same expression for the radiation field in the Fraunhofer region. In transmission through a homogeneous space, circular laser beam retains Gaussian shape regarding radial amplitude variation, but its phase front becomes curved and spherical at sufficiently far distance. Elliptical laser beam has to be considered independently along the main axes of the ellipse, and its amplitude field variation changes from elliptical to circular and then again to elliptical with the reversed axes. While the axial field of circular laser beam falls inversely with distance, it falls faster in the elliptical beam case.

2. CIRCULAR BEAM TRANSFORMATION THROUGH SPACE

Basic geometry used in calculation of circular laser beam is shown In Fig.1. In the aperture we assume the scalar field of the form

$$E(x,y,0) = E_0 \exp\left\{-\frac{x^2+y^2}{w_0^2}\right\} \tag{1}$$

where E_0 is the axial field intensity at $z=0$, and w_0 is the beam waist.

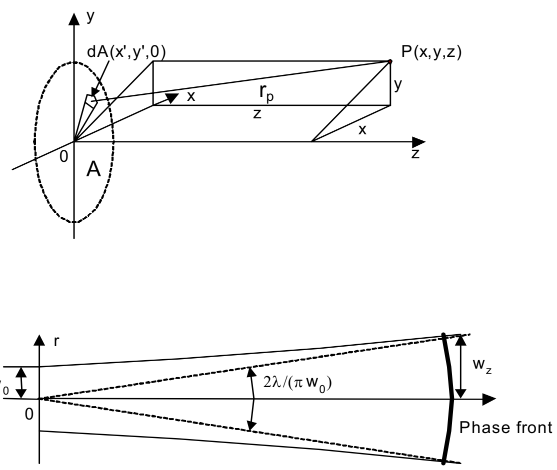

Fig.1. Geometry used in defining laser beam

Analytical expression for the beam at the plane $z = z_1$, can be expressed in the form

$$\begin{aligned} E_{z1}(x,y,z_1) &= \frac{w_0}{w_{z1}} E_0 \exp\left\{-\frac{x^2+y^2}{w_{z1}^2}\right\}\times \\ &\times \exp\left\{-j\left[kz_1 + \frac{\pi(x^2+y^2)}{\lambda R_{z1}} - \operatorname{arctg}\frac{2z_1}{kw_0^2}\right]\right\} \end{aligned} \tag{2}$$

where the beam radius and the radius of the phase front at $z = z_1$ are

$$w_{z1} = w_0\sqrt{1+\left(z_1^2/z_r^2\right)}, \qquad R_{z1} = z_1\left(1+z_r^2/z_1^2\right),$$

with $z_r = \pi w_0^2/\lambda$, $k = 2\pi/\lambda$, λ is the wavelength. At distances for which $z_1 >> z_r$,

$$w_{z1} \approx \frac{w_0 z_1}{z_r} = \frac{\lambda z_1}{\pi w_0}, \text{ and } R_{z1} \approx z_1 .$$

The far field is again Gaussian but with the complex parameters

$$\begin{aligned} E_{z1}(x,y,z_1) &= \frac{z_r}{z_1} E_0 \exp\left\{-\frac{x^2+y^2}{w_0^2}\left(\frac{z_r^2}{z_1^2} + j\frac{z_r}{z_1}\right)\right\}\times \\ &\times \exp\left(-j2\frac{z_1 z_r}{w_0^2} + j\frac{\pi}{2}\right) \end{aligned} \tag{3}$$

For a given laser z_r and w_0 are constants. If we take into account that $x^2 + y^2 = r^2$ we see that the far field beam is again circular and there is no ϕ angle dependence. The axial field amplitude decreases inversely with the distance. At a fixed distance $z = z_1$, the contours of constant amplitude and phase are circles.

3. CIRCULAR BEAM TRANSFORMATION THROUGH A LENS

The transformation of Gaussian beam through a converging or diverging lens can be obtained by simple calculations based on the *q* parameter binomial transformation [3]. However, if the lens radius is comparable with the beam radius significant errors are encountered and we need to solve a complex electromagnetic problem. Here we will show steps to be taken if the lens radius is arbitrary.

If at the plane $z=z_1$ we place a thin convex lens of focal distance *f*, it will affect only phase of the field which is possible to take into account by multiplying the field in Eq. (2) by the function:

$$\exp\left[j\frac{k}{2f}\left(x^2+y^2\right)\right] \tag{4}$$

For the sake of simplicity we can now assume to have a lens of infinite size so that the aperture field behind the lens can be expressed as the Kirchhoff's-Huygens integral. If the size of lens is finite then the integral limits have to be reduced and no closed form solution is possible.

$$e_{rad}(x,y,z) = \frac{j}{\lambda}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e_{lens}(x',y',z_1)\frac{e^{-jkr_p}}{r_p}dx'dy' \tag{5}$$

where $e_{lens}(x',y',z_1) = e_z(x',y',z_1)\exp\left(j\frac{k}{2f}(x'^2+y'^2\right)$ and $r_p \approx z\left[1+\frac{(x-x')^2+(y-y')^2}{2z^2}\right]$. In the expression for r_p new axial coordinate is measured from the aperture to the observation point behind the lens. In this approximation we can put $r_p = z$ in the denominator.

The integral (5) can be converted in the form suitable for integration. After some manipulations we reduce problem to solving the integral (shown only for the x')

$$\exp\left[-\left(\frac{k^2}{4a_x^2z^2}+j\frac{k}{2z}\right)x^2\right]\int_{-\infty}^{\infty}\exp\left[-\left(a_xx'-j\frac{k}{2a_xz}x\right)^2\right]dx' \tag{6}$$

where

$$a_x^2 = \frac{1}{w_{z1x}^2}+j\frac{k}{2}\left(\frac{1}{R_{z1x}}+\frac{1}{z}-\frac{1}{f}\right)$$

After substitution $u = a_xx' - j\frac{kx}{2a_xz}$, the integral in Eq.(6) can be solved and the final results is

$$\frac{\sqrt{\pi}}{a_x}\exp\left[-\left(\frac{k^2}{4a_x^2z^2}+j\frac{k}{2z}\right)x^2\right] \tag{7}$$

In a similar way the integral with respect to y' leads to

$$\frac{\sqrt{\pi}}{a_y}\exp\left[-\left(\frac{k^2}{4a_y^2z^2}+j\frac{k}{2z}\right)y^2\right] \tag{8}$$

The radiated field is proportional to the product of (7) and (8) and it is also Gaussian. After further calculation the results for the filed after an infinite lens is found to be identical with that obtained by the *q* parameters calculation. The effect of finite lens size can be found in ref.[

4. ELLIPTICAL BEAM TRANSFORMATION

The electromagnetic field radiated from an aperture illuminated by the field:

$$e_{ap}(x',y',0) = E_0\exp\left[-\left(\frac{x'^2}{w_{0x}^2}+\frac{y'^2}{w_{0y}^2}\right)\right] \tag{9}$$

where w_{0x}, w_{0y} are the waists of the Gaussian elliptical beam and E_0 is the electric field on the beam axis. The radiated field at some distance $z=z_1$ can be written in the form [5]:

$$e_z = E_0\sqrt{\frac{w_{0x}w_{0y}}{w_{z1x}w_{z1y}}}\exp\left[\begin{array}{l}-\left(\frac{1}{w_{z1x}^2}+j\frac{\pi}{\lambda R_{z1x}}\right)x^2-\\-\left(\frac{1}{w_{z1y}^2}+j\frac{\pi}{\lambda R_{z1y}}\right)y^2\end{array}\right]e^{[-jk(z_1-\psi)]}$$

$$\psi = \frac{1}{2}\tan^{-1}\left(\frac{\lambda z_1}{\pi w_{0x}^2}\right)+\frac{1}{2}\tan^{-1}\left(\frac{\lambda z_1}{\pi w_{0y}^2}\right) \tag{10}$$

If at the plane $z=z_1$ we have a thin convex lens of focal distance *f*, it will affect only phase of the field which is possible to introduce by multiplying the field in Eq. (10) by the function:

$$\exp\left[j\frac{k}{2f}\left(x^2+y^2\right)\right] \tag{11}$$

For the sake of simplicity we can now assume to have a lens of infinite size so that the aperture field behind the lens can be expressed as:

$$e_{lens}(x',y',z_1) = e_z(x',y',z_1)\ \exp\left[j\frac{k}{2f}\left(x'^2+y'^2\right)\right] \tag{12}$$

where again we used (x', y') as the aperture coordinates, but they are different from those used in Eq. (9).
To find the radiation field we apply the Kirchhoff's-Huygens integral

$$e_{rad}(x,y,z) = \frac{j}{\lambda}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e_{lens}(x',y',z_1)\frac{e^{-jkr_p}}{r_p}dx'dy' \quad (13)$$

where $r_p^2 = (x-x')^2 + (y-y')^2 + z^2$. Notice that the coordinate z is measured from the lens aperture so that we do not take into account the distance between the waists plane and the lens plane z_1.
Calculation of the double integral in (13) can be simplified if we assume that r_p in the denominator can be replaced by z, and in the exponential term by

$$r_p \approx z\left[1+\frac{(x-x')^2+(y-y')^2}{2z^2}\right] \quad (14)$$

Now the radiation field can be calculated from two line integrals as the integrand in this case can be written in the form of product of functions $e^{X(x')}e^{Y(y')}$, where

$$X(x') = -\left[\frac{1}{w_{z1x}^2}+j\frac{k}{2}\left(\frac{1}{R_{z1x}}+\frac{1}{z}-\frac{1}{f}\right)\right]x'^2 + j\frac{kx}{z}x' - j\frac{k}{2z}x^2 \quad (15)$$

$$X(y') = -\left[\frac{1}{w_{z1y}^2}+j\frac{k}{2}\left(\frac{1}{R_{z1y}}+\frac{1}{z}-\frac{1}{f}\right)\right]y'^2 + j\frac{ky}{z}y' - j\frac{k}{2z}y^2 \quad (16)$$

We can convert the above expressions in the form suitable for integration. After some manipulations that lead to the integral (shown only for the x')

$$\exp\left[-\left(\frac{k^2}{4a_x^2z^2}+j\frac{k}{2z}\right)x^2\right]\int_{-\infty}^{\infty}\exp\left[-\left(a_xx'-j\frac{k}{2a_xz}x\right)^2\right]dx' \quad (17)$$

where

$$a_x^2 = \frac{1}{w_{z1x}^2}+j\frac{k}{2}\left(\frac{1}{R_{z1x}}+\frac{1}{z}-\frac{1}{f}\right)$$

After substitution $u = a_xx' - j\frac{kx}{2a_xz}$, the integral in Eq.(17) can be solved and the final results is

$$\frac{\sqrt{\pi}}{a_x}\exp\left[-\left(\frac{k^2}{4a_x^2z^2}+j\frac{k}{2z}\right)x^2\right] \quad (18)$$

In a similar way the integral with respect to y' leads to

$$\frac{\sqrt{\pi}}{a_y}\exp\left[-\left(\frac{k^2}{4a_y^2z^2}+j\frac{k}{2z}\right)y^2\right] \quad (19)$$

where

$$a_y^2 = \frac{1}{w_{z1y}^2}+j\frac{k}{2}\left(\frac{1}{R_{z1y}}+\frac{1}{z}-\frac{1}{f}\right)$$

Finally

$$e_{rad}(x,y,z) = \frac{jE_{z1}}{\lambda\, za_xa_y}e^{-j(kz-\psi)}\times$$
$$\times\exp\left[-\left(\frac{k^2}{4a_x^2z^2}+j\frac{k}{2z}\right)x^2-\left(\frac{k^2}{4a_y^2z^2}+j\frac{k}{2z}\right)y^2\right] \quad (20)$$

The radiated field is also Gaussian and the 1/e half width at some distance z can be found after separating the real part of the term

$$\frac{k^2}{4z^2\left[\frac{1}{w_{z1x}^2}+j\frac{k}{2}\left(\frac{1}{R_{z1x}}+\frac{1}{z}-\frac{1}{f}\right)\right]} = \frac{\frac{k^2}{w_{z1x}^2}-j\frac{k}{2}\left(\frac{1}{R_{z1x}}+\frac{1}{z}-\frac{1}{f}\right)}{4z^2\left[\frac{1}{w_{z1x}^4}+\frac{k^2}{4}\left(\frac{1}{R_{z1x}}+\frac{1}{z}-\frac{1}{f}\right)^2\right]} \quad (21)$$

The required square of the 1/e half-widths are then

$$w_{zfx}^2 = \frac{4z^2w_{z1x}^2}{k^2}\left[\frac{1}{w_{z1x}^4}+\frac{k^2}{4}\left(\frac{1}{R_{z1x}}+\frac{1}{z}-\frac{1}{f}\right)^2\right] = w_{z1x}^2\left(1+\frac{z}{R_{z1x}}-\frac{z}{f}\right)^2+\frac{z^2\lambda^2}{\pi^2w_{z1x}^2} \quad (22)$$

$$w_{zfy}^2 = w_{z1y}^2\left(1+\frac{z}{R_{z1y}}-\frac{z}{f}\right)^2+\left(\frac{z\lambda}{\pi w_{z1y}}\right)^2 \quad (23)$$

For z=0 we have $w_{0fx} = w_{z1x}$ and $w_{0fy} = w_{z1y}$ as it should be.
The position of the beam waist after the lens is obtained from the derivative of (22) or (23). For the x-component it is

$$z_{\min(x)} = \frac{w_{z1x}^2\left(\frac{1}{f}-\frac{1}{R_{z1x}}\right)}{w_{z1x}^2\left(\frac{1}{f}-\frac{1}{R_{z1x}}\right)^2+\left(\frac{\lambda}{\pi w_{z1x}}\right)^2} \tag{24}$$

After substitution of w_{z1x} and R_{z1x} from Ref.[5] into Eq.(24), identical expression as the one obtained by the q-parameter calculation is found:

$$z_{\min(x)} = \frac{z_1(z_1-f)+a_{0x}^2}{(z_1-f)^2+a_{0x}^2} f \tag{25}$$

For the y-component similar calculation leads to

$$z_{\min(y)} = \frac{z_1(z_1-f)+a_{0y}^2}{(z_1-f)^2+a_{0y}^2} f \tag{26}$$

Upon substitution of the beam waist position from Eq.(24) into Eq.(22) it is found that the waist is

$$\frac{1}{w_{wx}^2} = \frac{1}{w_{z1x}^2}+\left(\frac{\pi w_{z1x}}{\lambda}\right)^2\left(\frac{1}{R_{z1x}}-\frac{1}{f}\right)^2 = \frac{(z_1-f)^2+a_{0x}^2}{w_{0x}^2 f^2} \tag{27}$$

for the amplitude at the axis one obtains which is identical to the one derived on the basis of q-parameter for the x-component. The y- component is obtained in a similar way and the only difference is the index in (27) "x" which has to be replaced by "y". After further elaborate calculations we find

$$|E_{rad}(x,y,z)| = \sqrt{\frac{w_{0x}w_{0y}}{w_{zfx}w_{zfy}}}E_0 \tag{28}$$

where w_{zfx} and w_{zfy} are obtained from Eq.(22) and (23), respectively.

5. LENS SYSTEMS

Lens system consisting of more than one lens is used to modify the waist of a Gaussian beam. Systems consisting of two converging lenses or diverging and converging lenses are called *the beam expander*. The scheme of two converging lenses is shown in figure below

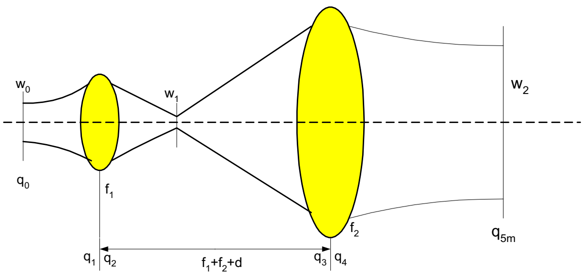

Fig.2 Two lens system

In investigation of two or more lens systems, matrix method of forming transfer matrix is advisable. Matrix of the system is formed by matrix product of individual sections as explained earlier. For example, the matrix two-lens system is obtained from the product of three matrices:

$$\begin{vmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{vmatrix}\begin{vmatrix} 1 & d \\ 0 & 1 \end{vmatrix}\begin{vmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{vmatrix} = \begin{vmatrix} 1-\frac{d}{f_1} & d \\ \frac{d}{f_1 f_2}-\frac{1}{f_1}-\frac{1}{f_2} & 1-\frac{d}{f_2} \end{vmatrix} \tag{29}$$

where f_1 and f_2 are the focal lengths of the first and the second lens to which the beam is incident. Distance between the lenses is d.

If the beam waist is at the distance z_1 from the first lens then the q factors are:

$q_0 = j\frac{\pi w_0^2}{\lambda} = jz_{a0}$, at the beam waist cross-section plane;

$q_1 = z_1 + jz_{a0}$, at the cross-section plane in front of the first lens;

q_2, at the cross-section after the first lens;

q_{2m}, at the cross-section of the beam waist after the first lens;

$q_3 = q_2 + d$, at the cross-section before the second lens, and

q_4, at the cross-section of the output plane of the second lens.

The parameter q_2 is obtained from

$$q_2 = \frac{q_1}{\frac{-q_1}{f_1}+1} \tag{30}$$

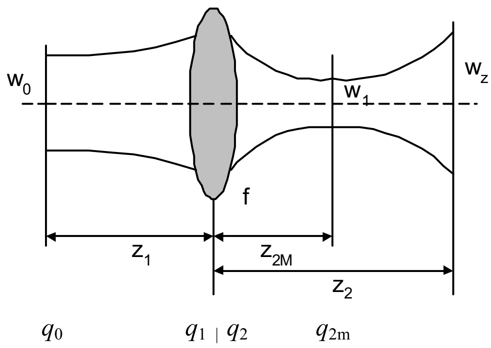

Fig.3. Beam transformation through a lens

The beam waist and its position after the first lens are obtained from $q_{2m} = q_2 + z_{2m}$, where $q_{2m} = j\frac{\pi w_1^2}{\lambda} = jz_{a1}$. Hence,

$$w_1 = w_0 \sqrt{\frac{\mathrm{Im}(q_2)}{\mathrm{Im}(q_0)}}, \qquad z_{2m} = -\mathrm{Re}(q_2) \qquad (31)$$

After substitution of defined parameters, we have also:

$$w_1 = \frac{w_0 f_1}{\sqrt{(z_1 - f_1)^2 + z_{a0}^2}}, \quad z_{2m} = \frac{z_1(z_1 - f_1) + z_{a0}^2}{(z_1 - f_1)^2 + z_{a0}^2} f_1 \qquad (32)$$

To find beam waist and its position after the two lenses we have to use the matrix of the whole system as given by (1) so that

$$q_4 = \frac{\left(1 - \frac{d}{f_1}\right) q_1 + d}{\left(\frac{d}{f_1 f_2} - \frac{1}{f_1} - \frac{1}{f_2}\right) q_1 + 1 - \frac{d}{f_2}} \qquad (33)$$

To find the beam parameter q_5 after the second lens we have simply: $q_5 = q_4 + z_5$. The new beam waist is obtained for a specific z_{5m}, where $q_{5m} = q_4 + z_{5m} = j z_{a3}$, i.e. the new beam waist after the system of two lenses is located at the plane where q_5 is imaginary. Since $z_{a2} = j\frac{\pi w_2^2}{\lambda}$, the new beam waist is obtained from:

$$w_2 = w_0 \sqrt{\frac{\mathrm{Im}(q_4)}{\mathrm{Im}(q_0)}} \qquad (34)$$

$$z_{5m} = -\mathrm{Re}(q_4) \qquad (35)$$

General solution of (34) and (35) is complex and will be given here in closed form only for effective magnification ratio:

$$M_d = \frac{M}{\sqrt{\left[1 + \frac{\delta}{f_1}\left(1 - \frac{z_1}{f_1}\right)\right]^2 + \left(\frac{\delta z_{a0}}{f_1^2}\right)^2}} \qquad (36)$$

where $M = f_2 / f_1$, is the maximum magnification, and $\delta = d - (f_1 + f_2)$.

In a special case when $z_1 = f_1$ and $d = f_1 + f_2$ the solution for (34) and (35) is simple:

$$w_2 = w_0 \frac{f_2}{f_1}, \text{ and } \quad z_{5m} = f_2 \qquad (37)$$

This case is referred as the astronomical beam expander if $f_2 > f_1$.

6. CONCLUSIONS

In transmission through various conditions in propagation, laser beam changes the beam waist and its position along the propagation axis. In paraxial cases, or in Fraunhofer region closed form expressions for the electromagnetic field can be found. If infinite space beam distribution is not applicable, or if finite sizes of lenses are present, more precise approach is needed and the solutions can be found by numerical means.

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